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WAYS OF ALLOWING FOR *A PRIORI* INFORMATION IN REGULARIZING GRADIENT ALGORITHMS

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Ways of allowing for *a priori* information on an unknown quantity in the solution of boundary-value and coefficient inverse problems of heat conduction by gradient methods are considered.

In the solution of inverse problems of heat conduction (IPHC), like any other ill-posed problem the qualitatively obtained approximations essentially depend on the proper and complete allowance for all the available *a priori* information about the solution being sought [1, 2]. And the widespread case in IPHC is the presence of information about the smoothness of the solution.

Let an IPHC be formulated as an operator equation of the first kind,

$$Au = f, \quad u \in U, \quad f \in F, \tag{1}$$

where we shall take the operator A as Frechet differentiable. The choice of the spaces U and F is dictated by the statement of the problem itself: They must contain sufficiently broad classes of functions, which will include all physically possible solutions u and any initial data f with allowance for the distortions introduced by the measurement systems. Therefore, the space L_2 of functions with an integrable square is taken most often as the spaces U and F. This is a Hilbert space, enabling one to apply gradient methods for the solution of Eq. (1).

For concrete problems, however, there is often additional, qualitative, *a priori* information about the solution being sought, which is usually given in one of two forms:

1) $u \in L[V]$, a transform of a certain continuous linear operator $L:V \to U$;

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2) u ∈ V, a certain Hilbert space with a stronger norm than in the space U. For example, if U = L_2, than Sobolev spaces W_2^k of functions having k generalized derivatives are often put forward as V.

We note that *a priori* information of these types alone is usually insufficient for the construction of regularizing algorithms, and serves only for narrowing the set of possible solutions. For the construction of regularizing gradient algorithms one can use, e.g., the discrepancy criterion, which is rigorously founded for linear ill-posed problems [3, 4], and also used successfully for the solution of certain nonlinear IPHC, those of [5, 6], for example.

To allow for information on smoothness of the first type one can convert to the auxiliary equation

$$Bv = f \quad B = AL: V \to F \tag{2}$$

and write for it the calculating equations of some gradient method:

$$v_{n+1} = v_n - \beta_n J' v_n = v_n - \beta_n (B')^*(Bv_n - f). \tag{3}$$

Here B' is the Frechet derivative of the operator B; $(B')^*$, operator conjugate with B'; $J'v_n$, gradient of the discrepancy functional of Eq. (2); β_n, step of the descent. In this case $(B')^* = ((AL)')^* = (A'L)^* = L^*(A'^*)$, since L' = L by virtue of the linearity of the operator L.

Applying the operator L to both sides of (3), we obtain a succession of approximations to the solution being sought:

$$u_{n+1} = u_n - \beta_n LL^* (A')^*(Au_n - f). \tag{4}$$

Such an approach is analyzed in more detail in [2].

The presence of information of the second type can be reduced to the case under consideration if one sets L = I, the operator of embedding of V into U, which sets each element u ∈ V in correspondence to the same element, but now treated in the space U. In this case the succession (4) takes the form

$$u_{n+1} = u_n - \beta_n I^* (A')^*(Au_n - f). \tag{5}$$

To construct the succession (5) one must obtain an expression for the operator I^*. Let us consider the particular case when $U = L_2[0, \tau_m]$, $V = W_2^k[0, \tau_m]$, $\|v\|^2_{W_2^k} = \|v\|^2_{L_2} + \rho \|v^{(k)}\|^2_{L_2}$, $\rho > 0$. By the definition of a conjugate operator, for any $u \in L_2$, $v \in W_2^k$, and $u^* = I^*u$ the identity $(u, v)_{L_2} = (u^*, v)_{W_2^k}$ must be satisfied. From this, through integration by parts, we have

$$(u, v)_{L_2} = \int_0^{\tau_m} u^* v d\tau + \rho \int_0^{\tau_m} u^{*(k)} v^{(k)} d\tau = \int_0^{\tau_m} u^* v d\tau + \rho u^{*(k)} v^{(k-1)}\Big|_0^{\tau_m} - \rho \int_0^{\tau_m} u^{*(k+1)} v^{(k-1)} d\tau = \ldots =$$

$$= \int_0^{\tau_m} [u^* + (-1)^k \rho u^{*(2k)}] v d\tau + \rho \sum_{j=1}^{k} u^{*(k+j-1)} v^{(k-j)}\Big|_0^{\tau_m}.$$

Thus, the function $u^* = I^*u$ is a solution of the boundary-value problem

$$u^*(\tau) + (-1)^k \rho u^{*(2k)}(\tau) = u(\tau),$$
$$u^{*(k)}(0) = u^{*(k)}(\tau_m) = \ldots = u^{*(2k-1)}(0) = u^{*(2k-1)}(\tau_m) = 0.$$

It is also easy to obtain an explicit expression for the function $u^*(\tau)$. For k = 1, for example,

$$u^*(\tau) = \frac{1}{\sqrt{\rho}} \frac{\mathrm{ch}\left(\frac{\tau}{\sqrt{\rho}}\right)}{\mathrm{sh}\left(\frac{\tau_m}{\sqrt{\rho}}\right)} \int_0^{\tau_m} \mathrm{ch}\left(\frac{\tau_m - \xi}{\sqrt{\rho}}\right) u(\xi) d\xi - \frac{1}{\sqrt{\rho}} \int_0^{\tau} \mathrm{sh}\left(\frac{\tau - \xi}{\sqrt{\rho}}\right) u(\xi) d\xi.$$

These approaches to the allowance for *a priori* information can be used in those cases when there are sufficiently simple expressions for the operator $(A')^*$, such as in boundary-value IPHC. In coefficient IPHC the expression for $(A')^*$ is inconvenient for calculation but, in return, the corresponding values for finite-dimensional approximations of the original operator A are easily calculated. Let us consider this case in more detail.

In the space U let a subspace U_m be chosen with a basis $\{\varphi_i\}$, $i = 1, \ldots, m$, and let the solution be sought in the form $u = \sum_{i=1}^{m} a_i\varphi_i \equiv \langle \bar{a}, \bar{\varphi} \rangle$. Then we consider a finite-dimensional approximation of Eq. (1):

$$A_m u = f,\ A_m: U_m \to F,\ A_m u = Au,\ u \in U_m. \tag{6}$$

If we introduce the operator $D\bar{a} = \langle \bar{a}, \bar{\varphi} \rangle$, then we can rewrite (6) in the form

$$C\bar{a} = f,\ C = A_m D: R^m \to F.$$

Let the expression for $(C')^*$ be known; then $C' = A'_m D' = A'_m D$, since D is a linear operator. Consequently, $(C')^* = D^*(A'_m)^*$. Hence $(A'_m)^* = (D^*)^{-1}(C')^*$, i.e., the expression for $(A'_m)^*$ can be obtained through $(C')^*$, and hence the expression for the discrepancy gradient of Eq. (6) can be obtained. For this we need to find $(D^*)^{-1}$. First we find D^* from the identity $(\bar{b}, \bar{b}^*)_{R^m} \equiv (D\bar{b}, u)$, $\bar{b} \in R^m$, $\bar{b}^* = D^* u$, $u = \langle \bar{a}, \bar{\varphi} \rangle \in U_m$. From this

$$(\bar{b}, \bar{b}^*)_{R^m} = \Big(\sum_{i=1}^{m} b_i\varphi_i, \sum_{j=1}^{m} a_j\varphi_j\Big)_U = \sum_{i=1}^{m} b_i \Big(\sum_{j=1}^{m} (\varphi_i, \varphi_j)_U a_j\Big) = (\bar{b}, G_U\bar{a})_{R^m},$$

where $G_U = [(\varphi_i, \varphi_j)_U]_m$ is the Gram matrix of the basis $\bar{\varphi}$. Consequently, $D^* u = G_U\bar{a}$. Since $(D^*)^{-1}D^* u = u = \langle \bar{a}, \bar{\varphi} \rangle$, we obviously have $(D^*)^{-1}\bar{b} = \langle G_U^{-1}\bar{b}, \bar{\varphi} \rangle$.

In order to allow for a priori information of the second type in the case under consideration, it is sufficient to take the matrix $G_V = [(\varphi_i, \varphi_j)_V]_m$ instead of the matrix G_U.

As an example, let us consider the problem of determining $u(T) = \lambda(T)$ from the known temperature $f(\tau) = T(0, \tau)$ at the point $x = 0$ of a body in which the process of heat transfer is described by the following equation and boundary conditions:

$$\begin{gathered} c(T)T_\tau = (u(T)T_x)_x,\ (x, \tau) \in \Omega = (0, b) \times (0, \tau_m], \\ T(x, 0) = \varphi_0(x),\ T(b, \tau) = T_b(\tau),\ -u(T)T_x|_{x=0} = q(\tau). \end{gathered} \tag{7}$$

The Frechet derivative of the operator of this problem $A'\Delta u = \Delta T(0, \tau)$ is found from the solution of the problem in increments,

$$\begin{gathered} (c\Delta T)_\tau = u\Delta T_{xx} + u_x\Delta T_x + (u_x\Delta T)_x + (T_x\Delta u)_x, \\ \Delta T(x, 0) = 0,\ \Delta T(b, \tau) = 0,\ \ -(u\Delta T)_x|_{x=0} = T_x\Delta u|_{x=0}. \end{gathered} \tag{8}$$

Then we consider the following conjugate problem for (8):

$$\begin{gathered} c\psi_\tau + u\psi_{xx} = 0, \\ \psi(x, \tau_m) = 0,\ \psi(b, \tau) = 0,\ -u\psi_x|_{x=0} = \Delta f(\tau). \end{gathered} \tag{9}$$

It is easy to verify that the following identity holds for any $\Delta u(x, \tau) \in L_2[\Omega]$, $\Delta f \in L_2[0, \tau_m]$:

$$\int_0^{\tau_m} \Delta T(0, \tau)\Delta f(\tau)\,d\tau \equiv \iint_\Omega \Delta u(x, \tau)[-T_x\psi_x]\,d\Omega. \tag{10}$$

If $\Delta u(T)$ is treated as an element of the space $L_2[T_0, T_m]$, then to determine $(A')^*$ it remains to perform a change of variables in the double integral in Eq. (10), introducing the new variables $T = T(x, \tau)$ and $w = w(x, \tau)$ such that the Jacobian $\frac{\partial(x,\tau)}{\partial(T,w)}$ is different from zero in Ω. Then, since $\Delta u(x,\tau) = \Delta u(T(x, \tau))$, we find

$$\iint_{\Omega} \Delta u\,[-T_x\psi_x]\,dxd\tau = \iint_{\Omega_1} \Delta u(T)[-T_x\psi_x]\left|\frac{\partial(x,\tau)}{\partial(T,w)}\right| dTdw = \int_{T_0}^{T} \Delta u(T)\,dT \int_{w_1(T)}^{w_2(T)} [-T_x\psi_x]\left|\frac{\partial(x,\tau)}{\partial(T,w)}\right| dw.$$

Consequently,

$$\Delta u^* = (A')^*\Delta f = \int_{w_1(T)}^{w_2(T)} [-T_x\psi_x]\left|\frac{\partial(x,\tau)}{\partial(T,w)}\right| dw.$$

It is difficult to construct a universal algorithm for calculating (A')* from this equation. But if we consider a finite-dimensional approximation of the problem in the form

$\Delta u(T) = \sum_{i=1}^{m} \Delta a_i \varphi_i(T)$, , then the necessity for a change of variables drops out. In this case from the identity (10) we obtain

$$\int_0^{\tau_m} \Delta T(0,\tau)\,\Delta f(\tau)\,d\tau = \sum_{i=1}^{m} \Delta a_i \iint_{\Omega} \varphi_i(T(x,\tau))[-T_x\psi_x]\,d\Omega = (\Delta\bar{a},\ \Delta\bar{a}^*)_{R^m}.$$

Consequently,

$$\Delta\bar{a}^* = (C')^*\Delta f = \left\{\iint_{\Omega} \varphi_1(T(x,\tau))[-T_x\psi_x]\,d\Omega,\ \ldots,\ \iint_{\Omega} \varphi_m[-T_x\psi_x]\,d\Omega\right\}.$$

Knowing (C')*, it is now easy, as was shown above, to find $(A'_m)^*$. As the functions $\varphi_i(T)$ one can choose B-splines of the corresponding orders: If one considers that $u(T)\in W_2^k[T_0, T_m]$, then one must take splines of an order no lower than k. The scalar products in the Gram matrix G_V are then calculated in the space $W_2^k[T_0, T_m]$.

NOTATION

U, F, V, Hilbert spaces; A', B', C', derivatives of the Frechet operators A, B, and C; U_m, m-dimensional subspace of the space U; $\bar{\varphi}$, basis in U_m; G_U, Gram matrix of the basis $\bar{\varphi}$; G_U^{-1}, inverse matrix for the matrix G_U; R^m, Euclidian space; λ(T), coefficient of thermal conductivity; C(T), coefficient of heat capacity; T, temperature; τ, time.

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